Quantum Potential and Random Phase-Space Dynamics

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Abstract

We investigate Euler-type local momentum conservation laws valid for various stochastic phase - space processes. The involved pressure contribution notoriously appears in a characteristic functional form $\pm\overrightarrow{\nabla}\left[2d^2(t)\frac{\triangle\rho^{1/2}}{\rho^{1/2}}\right]$ which stays in close affinity with the quantum mechanical notion of the de Broglie - Bohm potential and its role in the hydrodynamical formulation of quantum dynamics.

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In the standard hydrodynamical picture of the Schrödinger quantum dynamics, we encounter two local conservation laws: the continuity equation for the probability density $\rho(\overrightarrow{x},t)$ and the Euler - type equation, controlling the space-time dependence of the current velocity $\overrightarrow{v}(\overrightarrow{x},t)$, in the specific form $(\partial_t+\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v}=\frac{1}{m}\overrightarrow{F}-\overrightarrow{\nabla}Q_q$ where $Q_q=-\frac{\hbar^2}{2m^2}\frac{\triangle\rho^{1/2}}{\rho^{1/2}}$ is the familiar de Broglie - Bohm quantum potential, [1]. By $\overrightarrow{F}$ we indicate the force field acting upon particles which in the conservative case coincides with $-\overrightarrow{\nabla}V$ for a suitable potential $V(\overrightarrow{x})$, while the nonconservative case in our further discussion will be restricted to the Lorentz force example $\overrightarrow{F}=e(\overrightarrow{E}+\overrightarrow{v}\times\overrightarrow{B})$ for a charge e.

Probability density dependent pressure-type potentials of the above Q_q form (up to coefficient adjustments) are known to appear in the hydrodynamical description of the standard Brownian motion, especially in the large friction regime, when the spatial Smoluchowski diffusion process enters the scene. In particular, the free Brownian motion is known [2, 3] to induce the current velocity $\overrightarrow{v}=-D\frac{\overrightarrow{\nabla}\rho}{\rho}$ which obeys the continuity equation (that trivially yields $\partial_t\rho=D\triangle\rho$) and the local momentum conservation law $(\partial_t+\overrightarrow{v}\cdot\overrightarrow{\nabla})\overrightarrow{v}=-2D^2\overrightarrow{\nabla}\frac{\triangle\rho^{1/2}}{\rho^{1/2}}$ where D is the diffusion constant. Notice the negative sign on the right-hand-side of that equation, to be compared with the corresponding quantum mechanical law.

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This observation has much broader significance in view of its validity also in the presence of external force fields. Namely, if we call our attention back to the standard Ornstein-Uhlenbeck (dissipative) framework, then the Langevin equation for mass m particle in an external field of force (we consider a conservative case) $\overrightarrow{F} = \overrightarrow{F}(\overrightarrow{x}) = -\overrightarrow{\nabla} V$ reads:

$$\frac{d\overrightarrow{x}}{dt} = \overrightarrow{u} \tag{1}$$

$$\frac{d\overrightarrow{u}}{dt} = -\beta\overrightarrow{u} + \frac{\overrightarrow{F}}{m} + \overrightarrow{A}(t) \tag{2}$$

Random acceleration $\overrightarrow{A}(t)$ obeys the white noise statistics: $\langle A_i(s)\rangle = 0$ and $\langle A_i(s) A_j(s')\rangle = 2q\delta(s-s')\delta_{ij}$, where $i = 1, 2, 3$.

Since things are specialized to the *standard* Brownian motion, we know a priori that noise intensity is determined by a parameter $q = D\beta^2$ where $D = \frac{kT}{m\beta}$, while the friction parameter β is given by the Stokes formula $m\beta = 6\pi\eta a$. Consequently, the effect of the surrounding medium on the motion of the particle is described by two parameters: friction constant β and bath temperature T. Assumptions about the asymptotic (equilibrium) Maxwell-Boltzmann distribution and the fluid reaction upon the moving particle are here implicit, [4].

The resulting (Markov) phase - space diffusion process is completely determined by the transition probability density $P(\overrightarrow{x}, \overrightarrow{u}, t | \overrightarrow{x}_0, \overrightarrow{u}_0, t_0)$, which is typically expected to be a fundamental solution of the Kramers equation:

$$\frac{\partial P}{\partial t} + \overrightarrow{u}\nabla_{\overrightarrow{x}} P + \left[-\beta\overrightarrow{u} + \frac{\overrightarrow{F}}{m}\right]\nabla_{\overrightarrow{u}} P = q\nabla^2_{\overrightarrow{u}} P \tag{3}$$

The associated spatial Smoluchowski diffusion process with a forward drift $\overrightarrow{b}(\overrightarrow{x}) = \frac{\overrightarrow{F}}{m\beta}$ is analyzed in terms of increments of the normalized Wiener process $\overrightarrow{W}(t)$. The infinitesimal increment of the configuration (position) random variable $\overrightarrow{X}(t)$ reads:

$$d\overrightarrow{X}(t) = \frac{\overrightarrow{F}}{m\beta}dt + \sqrt{2D}d\overrightarrow{W}(t) \tag{4}$$

The related Fokker-Planck equation for the spatial probability density $\rho(\overrightarrow{x}, t)$ reads $\partial_t \rho = D\triangle\rho - \overrightarrow{\nabla}\cdot(\rho\overrightarrow{b})$ and explicitly employs the large friction regime, [5, 4, 6]. In fact, we take for granted that both time and space scales *of interest* (i.e. those upon which the accumulation of relevant random events prove to be significant) largely exceed the relaxation time interval β^{-1} and that dominant contributions "of interest" come from velocities $|\overrightarrow{u}| \leq (kT/m) = (q/\beta)^{1/2}$ and that the corresponding variation of $\overrightarrow{r}$ is sufficiently small (actually it is of the order $|\overrightarrow{u}|/\beta \equiv (q/\beta^3)^{1/2}$), [4].

Under those assumptions the Fokker -Planck equation for the spatial Markov process arises as the scaling ($\beta \gg 1$) limit of the 0-th order moment equation associated with the original Kramers law of random phase-space dynamics. In fact, by following the traditional pattern of hydrodynamical formalism, [7, 8], we infer the closed system of two (which is special to Markovian diffusions !) local conservation laws for the Smoluchowski process, [6, 9]:

$$\begin{aligned} \partial_t \rho + \overrightarrow{\nabla} \cdot (\overrightarrow{v} \rho) &= 0 \qquad (5) \\ (\partial_t + \overrightarrow{v} \cdot \overrightarrow{\nabla}) \overrightarrow{v} &= \overrightarrow{\nabla} (\Omega - Q) \, . \qquad (6) \end{aligned}$$

Here (we use a short-hand notation $\overrightarrow{v}(\overrightarrow{x}, t) \doteq \overrightarrow{v}$)

$$\overrightarrow{v}(\overrightarrow{x}, t) = \frac{\overrightarrow{F}}{m\beta} - D \frac{\overrightarrow{\nabla} \rho}{\rho} \qquad (7)$$

defines so-called current velocity of Brownian particles and, when inserted to Eq. (5), transforms the continuity equation into the Fokker-Planck equation, [5].

Eq. (6) stands for the scaling limit of the first order moment equation and directly corresponds to the familiar Euler equation, characterizing the momentum conservation law in lowest order approximation of kinetic theory based on the Boltzmann equation, [8]. However, the large friction regime enforces here a marked difference in the local momentum conservation law, in comparison with the standard Euler equation for a nonviscous fluid or gas. Namely, instead of the kinetic theory motivated expression for e.g. rarified gas:

$$(\partial_t + \overrightarrow{v} \cdot \overrightarrow{\nabla}) \overrightarrow{v} = \frac{\overrightarrow{F}}{m} - \frac{\overrightarrow{\nabla} P}{\rho} \qquad (8)$$

where $P(\overrightarrow{x})$ stands for the pressure function (to be fixed by a suitable equation of state) and $\overrightarrow{F}$ is the very same (conservative $-\overrightarrow{\nabla} V$) force acting upon particles as that appearing in the Kramers equation (3), the Smoluchowski regime (6) employs the emergent volume force (notice the positive sign) $+\overrightarrow{\nabla} \Omega$ instead of $-\overrightarrow{\nabla} V$:

$$\Omega = \frac{1}{2} \left(\frac{\overrightarrow{F}}{m\beta} \right)^2 + D \overrightarrow{\nabla} \cdot \left(\frac{\overrightarrow{F}}{m\beta} \right) \qquad (9)$$

and the pressure-type contribution $-\overrightarrow{\nabla} Q$ where, [6] (see also [2, 3])

$$Q = 2D^2 \frac{\triangle \rho^{1/2}}{\rho^{1/2}} \qquad (10)$$

and $\triangle = \overrightarrow{\nabla}^2$ is the Laplace operator, does not leave any room for adidtional constraints upon the system (like e.g. the familiar equation of state).

To have a glimpse of a dramatic difference between physical messages conveyed respectively by equations (6) and (8), it is enough to insert in (8) the standard equation of state $P(\vec{x}) = \alpha\rho^{\beta}$ with $\alpha, \beta > 0$ and choose $\vec{F} = -\omega^2 \vec{x}$ to represent the harmonic attraction in Eqs. (2) - (9), see also [6].

Markovian diffusion processes with the inverted sign of $\vec{\nabla}(\Omega - Q)$ in the local momentum conservation law (6) i. e. respecting

$$(\partial_t + \vec{v} \cdot \vec{\nabla})\vec{v} \quad = \quad \vec{\nabla}(Q - \Omega) \tag{11}$$

instead of Eq. (6), were considered in Ref. [6] as implementations of the "third Newton law in the mean". Nonetheless, also under those premises, the volume force term $-\vec{\nabla}\Omega$ in Eq. (11) does *not* in general coincide with the externally acting conservative force contribution (e.g. acceleration) $\frac{1}{m}\vec{F} = -\frac{1}{m}\vec{\nabla}V$ akin to Eq. (8).

Accordingly, the effects of external force fields acting upon particles are significantly distorted while passing to the local conservation laws in the large friction (Smoluchowski) regime.

That becomes even more conspicuous in case of the Brownian motion of a charged particle in the constant magnetic field. In the Smoluchowski (large friction) regime, friction completely smoothes out any rotational (due to the Lorentz force) features of the process. In the corresponding local momentum conservation law there is *no* volume force contribution at all and merely the "pressure-type" potential Q appears in a rescaled form, [10]:

$$Q = \frac{\beta^2}{\beta^2 + \omega_c^2} \cdot 2D^2 \frac{\triangle \rho^{1/2}}{\rho^{1/2}} \tag{12}$$

where β is the (large) friction parameter and $\omega_c = \frac{eB}{m}$ is the rotational frequency of the charge q_e particle in a constant homogeneous magnetic field $\vec{B} = (0, 0, B)$. Clearly, for moderate frequency values ω_c (hence the magnetic field intensity) and sufficiently large β even this minor scaling remnant of the original Lorentz force would effectively disappear, yielding Eq. (10).

This observation is to be compared with results of Refs. [11] where frictionless stochastic processes were invoked to analyze situations present in magnetospheric environments. Specifically, one deals there with charged particles in a uniform magnetic field which experience stochastic electrical forcing. In the absence of friction, the rotational Lorentz force input should clearly survive when passing to the local conservation laws, in plain contrast with the Smoluchowski regime. By disregarding friction it is also possible to reproduce exactly the conservative external force acting upon particles in the local conservation laws, as originally suggested by [5].

Therefore, let us consider the frictionless phase - space dynamics in some detail. The Langevin-type equation for a particle which is not suffering any friction while being subject to random acceleration reads:

$$\begin{aligned} \frac{dx}{dt} &= u \\ \frac{du}{dt} &= A(t)\ . \end{aligned} \tag{13}$$

We fix initial conditions: $x(t_0) = x_0,\ u(t_0) = u_0,\ t_0 = 0$. The fluctuating term $A(t)$ is assumed to display a standard white noise statistic with $A(t)$ being independent of u and the noise intensity parameter $q > 0$ left unspecified.

This random dynamics induces a transition probability density $P(x, u, t|\, x_0, u_0, t_0)$ which uniquely defines the corresponding time homogeneous phase-space Markovian diffusion process executed by (x, u). Here, the function $P(x, u, t|\, x_0, u_0, t_0)$ is the fundamental solution of the Fokker-Planck equation associated with the above Langevin equation:

$$\frac{\partial P}{\partial t} = -u\frac{\partial P}{\partial x} + q\frac{\partial^2 P}{\partial u^2} \tag{14}$$

A fundamental solution of Eq. (14) was first given by Kolmogorov [12]:

$$P(x, u, t|\, x_0, u_0, t_0 = 0) = \frac{1}{2\pi}\frac{\sqrt{12}}{2qt^2}\exp\left[-\frac{(u-u_0)^2}{4qt} - \frac{3\left(x - x_0 - \frac{u+u_0}{2}t\right)^2}{qt^3}\right]. \tag{15}$$

We are interested in passing to a hydrodynamical picture, following the traditional recipes [7, 8]. To that end we need to propagate certain initial probability density and investigate effects of the random dynamics. Let us choose most obvious (call it natural) example of:

$$\rho_0(x, u) = \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}}\exp\left(-\frac{(x - x_{ini})^2}{2a^2}\right)\left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}}\exp\left(-\frac{(u - u_{ini})^2}{2b^2}\right). \tag{16}$$

so that at time t we have $\rho(x, u, t) = \int P(x, u, t|\, x_0, u_0, t_0 = 0)\,\rho_0(x_0, u_0)\,dx_0 du_0$.

Since $P(x, u, t|\, x_0, u_0, t_0)$ is the fundamental solution of the Kramers equation, the joint density $\rho(x, u, t)$ is also the solution and can be written in the familiar, [4], form of

$$W(R, S) = \left(\frac{1}{4\pi^2(fg - h^2)}\right)^{\frac{1}{2}}\exp\left[-\frac{gR^2 - 2hRS + fS^2}{2(fg - h^2)}\right] \tag{17}$$

for $\rho(x, u, t) = W(R, S)$. However, in the present case functional entries are adopted to the frictionless motion and read as follows:

$$S = u - u_{ini}$$

$$R = x - x_{ini} - u_{ini}t$$

$$f = a^2 + b^2 t^2 + \frac{2}{3} q t^3$$

$$g = b^2 + 2qt \tag{18}$$

$$h = b^2 t + q t^2 \,.$$

The marginals $\rho(x,t) = \int \rho(x,u,t)\, du$ and $\rho(u,t) = \int \rho(x,u,t)\, dx$ are

$$\rho(x,t) = \left(\frac{1}{2\pi f}\right)^{\frac{1}{2}} \exp\left(-\frac{R^2}{2f}\right) = \left(\frac{1}{2\pi\left(a^2 + b^2 t^2 + \frac{2}{3} q t^3\right)}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini} - u_{ini} t)^2}{2\left(a^2 + b^2 t^2 + \frac{2}{3} q t^3\right)}\right) \tag{19}$$

and

$$\rho(u,t) = \left(\frac{1}{2\pi g}\right)^{\frac{1}{2}} \exp\left(-\frac{S^2}{2g}\right) = \left(\frac{1}{2\pi\left(b^2 + 2qt\right)}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2\left(b^2 + 2qt\right)}\right) \tag{20}$$

Let us introduce an auxiliary (reduced) distribution

$$\widetilde{W}(S|R) = \frac{W(S,R)}{\int W(S,R)\, dS} = \left(\frac{1}{2\pi\left(g - \frac{h^2}{f}\right)}\right)^{\frac{1}{2}} \exp\left(-\frac{\left|S - \frac{h}{f} R\right|^2}{2\left(g - \frac{h^2}{f}\right)}\right) \tag{21}$$

where in the denominator we recognize the marginal spatial distribution $\int W(S,R)\, dS \doteq w$ of $W(S,R)$.

Following the standard hydrodynamical picture method [7, 8, 9] we define local (configuration space conditioned) moments: $\langle u \rangle_x = \int u \widetilde{W} du$ and $\langle u^2 \rangle_x = \int u^2 \widetilde{W} du$. From (21) it follows that

$$\langle u \rangle_x = u_{ini} + \frac{h}{f} R = u_{ini} + \frac{b^2 t + q t^2}{a^2 + b^2 t^2 + \frac{2}{3} q t^3} \left[x - x_{ini} - u_{ini} t\right] \tag{22}$$

$$\langle u^2 \rangle_x - \langle u \rangle_x^2 = \left(g - \frac{h^2}{f}\right) = \frac{q\, t^3\, \left(2\, b^2 + q\, t\right) + 3\, a^2\, \left(b^2 + 2\, q\, t\right)}{3\, a^2 + t^2\, \left(3\, b^2 + 2\, q\, t\right)} \tag{23}$$

The first two moment equations for the Kramers equation are easily derivable. Namely, the continuity (0-th moment) and the momentum conservation (first moment) equations come out in the form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x}\left(\langle u \rangle_x w\right) = 0 \tag{24}$$

$$\frac{\partial}{\partial t}\left(\langle u \rangle_x w\right) + \frac{\partial}{\partial x}\left(\langle u^2 \rangle_x w\right) = 0 \tag{25}$$

These equations yield the local momentum conservation law in the familiar form

$$\left(\frac{\partial}{\partial t} + \langle u\rangle_x \frac{\partial}{\partial x}\right)\langle u\rangle_x = -\frac{1}{w}\frac{\partial P_{kin}}{\partial x} \tag{26}$$

where we encounter the standard [8] textbook notion of the pressure function

$$P_{kin}(x,t) = \left[\left\langle u^2\right\rangle_x - \langle u\rangle_x^2\right] w(x,t)\ . \tag{27}$$

The marginal density w obeys $\frac{\nabla w}{w} = -2f\nabla\left[\frac{\Delta w^{1/2}}{w^{1/2}}\right]$ and that in turn implies

$$-\frac{1}{w}\frac{\partial P_{kin}}{\partial x} = 2\left(fg - h^2\right)\nabla\left[\frac{\Delta w^{1/2}}{w^{1/2}}\right]\ . \tag{28}$$

As a consequence, the local conservation law (26) takes the form

$$\left(\frac{\partial}{\partial t} + \langle u\rangle_x \nabla\right)\langle u\rangle_x = -\frac{\nabla P_{kin}}{w} = +2\left(fg - h^2\right)\nabla\left[\frac{\Delta w^{1/2}}{w^{1/2}}\right] \doteq +\nabla Q \tag{29}$$

where (we point out the plus sign in the above, see e.g. Eq. (11))

$$fg - h^2 = a^2b^2 + 2a^2qt + \frac{2}{3}b^2qt^3 + \frac{1}{3}q^2t^4 \doteq D^2(t) \tag{30}$$

and by adopting the notation $D^2(t) \doteq fg - h^2$ we get $-\frac{1}{w}\frac{\partial P_{kin}}{\partial x} = +\overrightarrow{\nabla}Q$ with the functional form of $Q(\overrightarrow{x},t)$ given by Eq. 10. Here, instead of a diffusion constant D we insert the (positive) time -dependent function $D(t)$.

With those notational adjustments, we recognize in Eq. (29) a consistent Euler form of the local momentum conservation law, in case of vanishing volume forces (c.f. Eqs. (6), (8), (11) for comparison).

A carefully executed, tedious calculation allows to demonstrate, [13], that an analogous result holds true in case of a harmonic attraction and for a nonconservative example of the Lorentz force in action. Both volume forces appear undistorted (that was *not* the case in the large friction regime) in the corresponding local momentum conservation laws. Indeed, we recover a universal relationship:

$$\left[\partial_t + \langle\overrightarrow{u}\rangle_{\overrightarrow{x}} \cdot \overrightarrow{\nabla}\right]\langle\overrightarrow{u}\rangle_{\overrightarrow{x}} = \frac{\overrightarrow{F}}{m} + 2d^2(t)\overrightarrow{\nabla}\left[\frac{\Delta w^{1/2}}{w^{1/2}}\right] \tag{31}$$

where $\overrightarrow{F}$ denotes external force acting on the particle, and $d = (\det C)^{\frac{1}{n}}$ where C is the covariance matrix of random variables (vectors) $\overrightarrow{S}$ and $\overrightarrow{R}$ (defined for each system) and n stands for the dimension of configuration space of appropriate system.

1. free particle: $\overrightarrow{R} = x$, $F \equiv 0$, $n = 1$

2. charged particle in a constant magnetic field: $\overrightarrow{R} = (x, y)$, $\overrightarrow{F} = e \langle \overrightarrow{u} \rangle_{\overrightarrow{x}} \times \overrightarrow{B}$, $n = 2$

3. harmonically bound particle: $\overrightarrow{R} = x$, $F = -m\omega^2 x$, $n = 1$

In case of harmonic and magnetic confinement, we need to have identified parameter range regimes that allow for a positivity of the time dependent coefficient $d(t) \doteq D(t)$ (in the force-free case it is positive with no reservations), in the pressure-type contribution acquiring a characteristic form of $-\frac{\overrightarrow{\nabla} \cdot \overleftrightarrow{P}}{w} = +\overrightarrow{\nabla} Q$. Indeed, only by means of a proper balance between q and ω_c we can achieve a positivity of the coefficient denoted formally $d^2(t)$ in case of the charged particle in a magnetic field:

$$d^2(t) = fg - h^2 - k^2 = \tag{32}$$
$$a^2\, b^2 \; - \frac{8\, q^2}{\omega_c^4} + \frac{4\, b^2\, q\, t}{\omega_c^2} + \frac{4\, q^2\, t^2}{\omega_c^2} + 2\, a^2\, q\, t + \frac{8\, q^2}{\omega_c^4}\, \cos(t\, \omega_c) - \frac{4\, b^2\, q}{\omega_c^3}\, \sin(t\, \omega_c)\,.$$

The time-dependent coefficient in case of the harmonic attraction reads:

$$d^2(t) = fg - h^2 = \tag{33}$$
$$\frac{-q^2 + 2\, b^2\, q\, t\, \omega^2 + 2\, q^2\, t^2\, \omega^2 + 2\, a^2\, b^2\, \omega^4 + 2\, a^2\, q\, t\, \omega^4 + q^2\, \cos(2\, t\, \omega) + q\, \omega \left(-b^2 + a^2\, \omega^2\right)\, \sin(2\, t\, \omega)}{2\, \omega^4}$$

and a proper balance between q and ω needs to be maintained to secure a positivity of the formal notion $d^2(t)$.

In Ref. [13] we have investigated the above expressions in the low noise intensity regime and for rather short duration times of the pertinent stochastic processes. That was motivated by the major conceptual input of [11, 13] that undamped random flights in external force fields may have physical relevance when dissipative time scales are much longer than the time duration of processes of interest, including the particle life-time.

Under those premises, we can view the noise intensity parameter q as a book-keeping label and investigate leading terms in all expressions encompassing both small q and short time t effects in the hitherto considered random dynamics.

In particular, it is obvious that by neglecting all q-dependent terms we readily arrive at the leading contribution in: $d^2(t) \to a^2 b^2 \doteq D^2$. By adopting this notation, Eq. (31) acquires a conspicuous quantum mechanical form [1] in the leading order of the q-dependent series expansion.

To elucidate the previous observation, let us explicitly associate the $q \ll 1$ (combined with small t) local momentum conservation law with that known to be appropriate for the quantum harmonic oscillator.

We impose the following condition on the dispersion parameters $a = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $b = \sqrt{\langle v^2 \rangle - \langle v \rangle^2}$ of initial distributions $\rho_0(x)$ and $\rho_0(v)$ respectively

$$a^2\, b^2 = \left(\frac{\hbar}{2m}\right)^2 . \tag{34}$$

This condition is obviously equivalent to imposing a priori the Heisenberg uncertainty relation $a\,(mb) = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\sqrt{\langle mv^2\rangle - \langle mv\rangle^2} = \frac{\hbar}{2}$ to be valid at an initial time instant (c.f. Eq. (16)). Upon an additional demand $b^2 = a^2\omega^2$ which is an identity for the particular choice of $b^2 = \frac{\hbar\omega}{2m}$, $a^2 = \frac{\hbar}{2m\omega}$, we recover

$$\rho\,(x,t) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{m\omega}{\hbar}\,(x - x_{ini}\cos\omega t)^2\right) \tag{35}$$

which directly corresponds to the quantum evolution of the coherent state of the harmonic oscillator.

To conclude, let us point out that neither large friction nor frictionless cases may be regarded as valid probabilistic phase-space motion counterparts of the quantum Schrödinger picture dynamics. However, only the frictionless case appears to be sufficiently close (at least in the small q and short duration time t regime) to the quantum dynamics in its hydrodynamical description and may be analyzed as a reliable approximation of the true (quantum !) state of affairs. The large friction dynamics, even if augmented by the concept of the "Brownian recoil principle", [14], in incapable of reproducing correctly the external force effects in the local momentum conservation laws. In Ref. [13] alternative physical mechanisms were proposed to enable a deeper affinity of the frictionless random dynamics to the quantum motion in the Schrödinger picture.

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